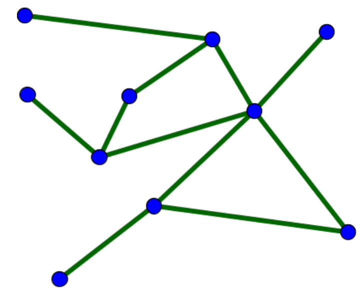
LECTURE 2

The Stochastic Ising Model (SIM)

§ SIM ON GRAPHS



Let $G = (V, E)$ be a finite connected non-oriented graph. Ising spins are attached to the vertices V and interact with each other along the edges E.

1. The energy associated with the configuration $\sigma = (\sigma_i)_{i \in V} \in \Omega = \{-1, +1\}^V$ is given by the Hamiltonian

$$H(\sigma) = -J \sum_{(i,j) \in E} \sigma_i \sigma_j - h \sum_{i \in V} \sigma_i$$

where $J > 0$ is the ferromagnetic interaction strength and $h > 0$ is the external magnetic field.

2. Spins flip according to Glauber dynamics $(\sigma_t^G)_{t \geq 0}$,

$$\forall \sigma \in \Omega \; \forall j \in V: \quad \sigma \to \sigma^j \text{ at rate } \mathrm{e}^{-\beta[H(\sigma^j)-H(\sigma)]_+}$$

where σ^j is the configuration obtained from σ by flipping the spin at vertex j, and $\beta > 0$ is the inverse temperature.

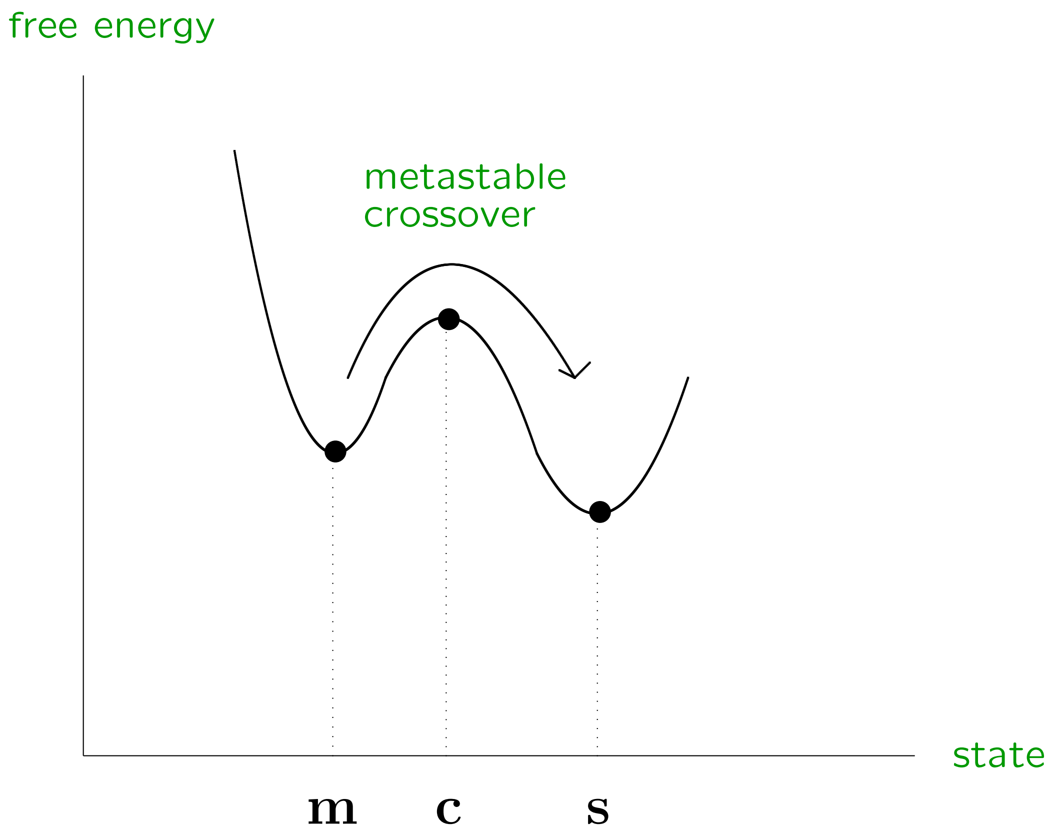
3. The Gibbs measure

$$\mu(\sigma) = \frac{1}{\Xi} \mathrm{e}^{-\beta H(\sigma)}, \qquad \sigma \in \Omega,$$

is the reversible equilibrium of this dynamics.

4. Three sets of configurations play a central role:

$$\mathbf{m} = \text{metastable state}$$
$$\mathbf{c} = \text{crossover state}$$
$$\mathbf{s} = \text{stable state.}$$



Caricature picture of the free energy landscape

[free energy = energy − entropy]

FORMAL DEFINITIONS:

(a) The stable state is the set of configurations having minimal energy:

$$\mathbf{s} = \Big\{ \sigma \in \Omega \colon\ H(\sigma) = \min_{\zeta \in \Omega} H(\zeta) \Big\}.$$

(b) The metastable state is the set of configurations not in $\mathbf{s}$ that lie at the bottom of the next deepest valley:

$$\mathbf{m} = \Big\{ \sigma \in \Omega \setminus \mathbf{s} \colon\ V_\sigma = \max_{\zeta \in \Omega \setminus \mathbf{s}} V_\zeta \Big\}$$

with V_ζ the minimal amount a path from ζ needs to climb in energy in order to reach an energy $< H(\zeta)$.

(c) The crossover state $\mathbf{c}$ is the set of configurations realising the min-max for paths connecting $\mathbf{m}$ and $\mathbf{s}$.

§ SIM ON THE COMPLETE GRAPH

Let us see what happens on the complete graph with N vertices. This is a mean-field setting.

The ferromagnetic interaction strength is chosen to be $J = N^{-1}$. It can be shown that the empirical magnetisation

$$m_t^N = \frac{1}{N} \sum_{i \in [N]} (\sigma_t^N)_i$$

performs a continuous-time random walk on the $2N^{-1}$-grid in $[-1,+1]$, in a potential that is given by the finite-volume free energy per vertex

$$f_{\beta,h}^N(m) = -\tfrac{1}{2}m^2 - hm + \beta^{-1} I^N(m)$$

with an entropy term

$$I^N(m) = -\frac{1}{N} \log \binom{N}{\frac{1+m}{2}N}.$$

In the limit $N \to \infty$, the empirical magnetisation performs a Brownian motion on $[-1,+1]$, in a potential that is given by the infinite-volume free energy per vertex

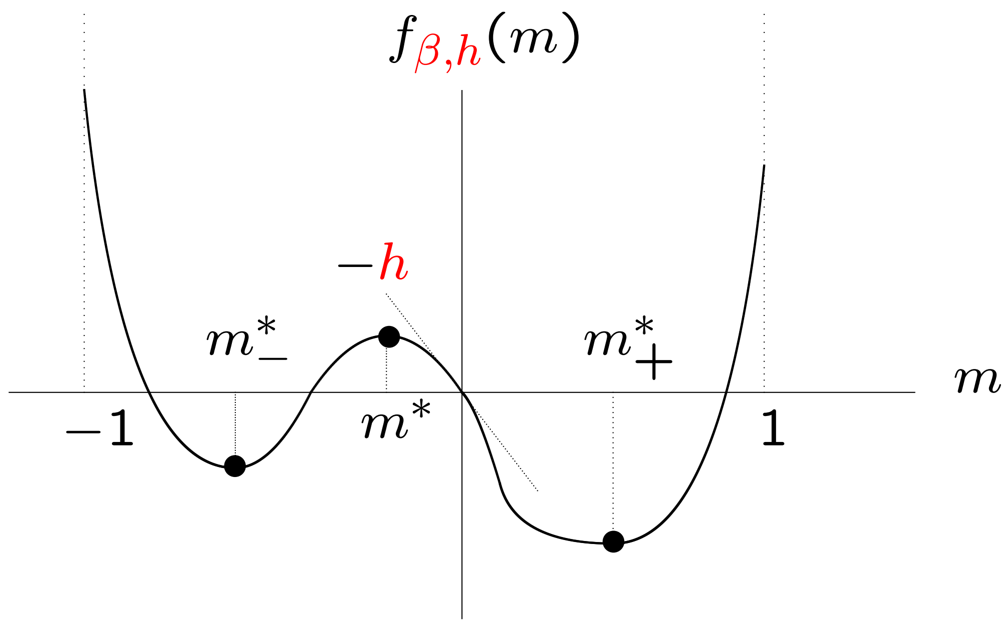
$$f_{\beta,h}(m) = -\tfrac{1}{2}m^2 - hm + \beta^{-1} I(m)$$

with

$$I(m) = \tfrac{1}{2}(1+m)\log(1+m) + \tfrac{1}{2}(1-m)\log(1-m),$$

where a redundant shift by $-\log 2$ is dropped.

The above formulas describe what is called the Curie-Weiss model with Glauber dynamics.



The free energy per vertex $f_{\beta,h}(m)$ at magnetisation m
(caricature picture with $\mathbf{m} = m^*_-$, $\mathbf{c} = m^*$, $\mathbf{s} = m^*_+$).

THEOREM 2.1 Bovier, Eckhoff, Gayrard, Klein 2000

If $\beta > 1$ *and* $h \in (0, \chi(\beta))$*, then*

$$\mathbb{E}^{\mathrm{CW}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+}) = K\, \mathrm{e}^{N\Gamma}[1 + o(1)], \qquad N \to \infty,$$

where $\mathbf{m}_N^-, \mathbf{m}_N^+$ *are sets of configurations for which the discrete magnetisations tends to the continuum magnetisations* m_-^*, m_+^**,*

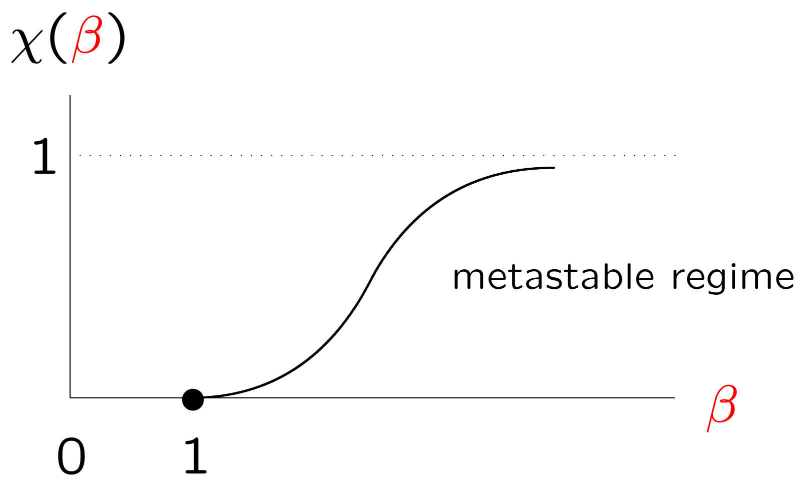
$$\Gamma = \beta\,[f_{\beta,h}(m^*) - f_{\beta,h}(m_-^*)]$$

$$K = \pi\beta^{-1}\sqrt{\frac{1+m^*}{1-m^*}\,\frac{1}{1-m_-^{*2}}\,\frac{1}{[-f''_{\beta,h}(m^*)]\,f''_{\beta,h}(m_-^*)}}$$

and

$$\chi(\beta) = \sqrt{1-\tfrac{1}{\beta}} - \tfrac{1}{2\beta}\log\left[\beta\left(1+\sqrt{1-\tfrac{1}{\beta}}\right)^2\right].$$

The conditions on β, h guarantee that $f_{\beta,h}$ has a double-well shape and represents the parameter regime for which metastable behaviour occurs.



The expression for the average crossover time in Theorem 2.1 is called the Kramers formula.

§ SIM ON RANDOM GRAPHS

The goal of this lecture is to investigate what can be said when the complete graph is replaced by a random graph.

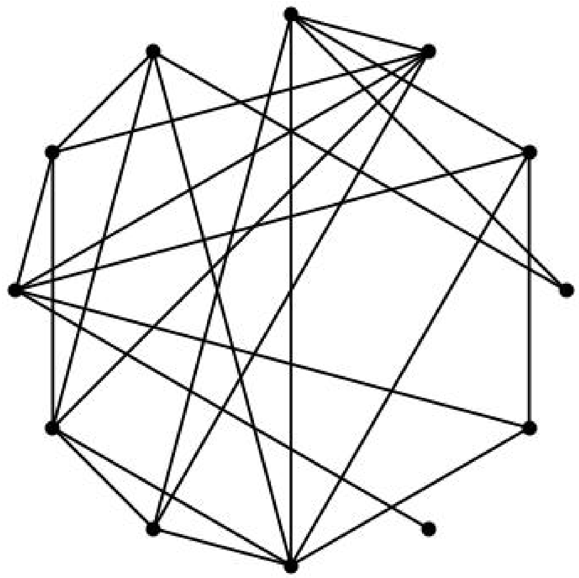
Our target will be to derive Arrhenius laws, i.e.,

$$\mathbb{E}_{\mathbf{m}}[\tau_{\mathbf{s}}] = K\,\mathrm{e}^{N\Gamma}[1+o(1)], \qquad N \to \infty,\ \beta \text{ fixed},$$

$$\mathbb{E}_{\mathbf{m}}[\tau_{\mathbf{s}}] = K\,\mathrm{e}^{\beta\Gamma}[1+o(1)], \qquad \beta \to \infty, N \text{ fixed}.$$

In general Γ, K are random and are hard to identify. In fact, in what follows we will mostly have to content ourselves with bounds on these quantities and with convergence in probability under the law of the random graph.

§ SIM ON THE ERDŐS-RÉNYI RANDOM GRAPH



Erdős-Rényi random graph: edge percolation

Take the complete graph with N vertices and retain edges with probability $p \in (0, 1)$.

THEOREM 2.2 den Hollander, Jovanovski 2021

On the Erdős-Rényi random graph with N *vertices, for* $J = 1/pN$*,* $\beta > 1$ *and* $h \in (0, \chi(\beta))$*,*

$$\mathbb{E}^{\mathrm{ER}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+}) = N^{E_N}\, \mathbb{E}^{\mathrm{CW}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+}), \qquad N \to \infty,$$

where E_N *is a random exponent that satisfies*

$$\lim_{N\to\infty} \mathbb{P}_{\mathrm{ER_N(p)}}\Big(|E_N| \leq \tfrac{11}{6}\tfrac{\beta}{p}(m^* - m_-)\Big) = 1,$$

with $\mathbb{P}_{\mathrm{ER_N(p)}}$ *the law of the random graph.*

Apart from a polynomial error term, the crossover time is the same on the Erdős-Rényi random graph as on the complete graph, after the change of interaction from $J = 1/N$ to $J = 1/pN$.

The asymptotic estimate of the crossover time is uniform in the starting configuration drawn from the set $\mathbf{m}_N^-$.

Note that J needs to be scaled up by a factor $1/p$ in order to allow for a comparison with the Curie-Weiss model: in the Erdős-Rényi model every spin interacts with $\sim pN$ spins rather than N spins. The critical value in equilibrium changes from 1 to $1/p$:

Bovier, Gayrard 1993

> On the complete graph the prefactor is constant and computable.
>
> On the Erdős-Rényi random graph it is random and more involved.

ENTROPY

The proof of Theorem 2.2 follows the pathwise approach to metastability.

In particular, the empirical magnetisation $(m_t^N)_{t \geq 0}$ is monitored on a mesoscopic space-time scale. The difficulty is that the lumping technique typical for mean-field settings is no longer available: after projection the Markov property is lost.

The way around this problem is via coupling: sandwich $(m_t^N)_{t \geq 0}$ between two Curie-Weiss models with a perturbed magnetic field h^N, tending to h as $N \to \infty$. The computations are rather elaborate and are beyond the scope of the present mini-course.

§ REFINEMENT OF THE PREFACTOR

THEOREM 2.3 Bovier, Marello, Pulvirenti 2021

For $\beta > 1$, $h > 0$ *small enough and* $s > 0$*,*

$$\lim_{N\to\infty} \mathbb{P}_{\mathrm{ER}_N(p)}\left(C_1 \mathrm{e}^{-s} \le \frac{\mathbb{E}^{\mathrm{ER}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+})}{\mathbb{E}^{\mathrm{CW}}_{\mathbf{m}_N^-}(\tau_{\mathbf{m}_N^+})} \le C_2 \mathrm{e}^{s} \right) \ge 1 - k_1 \mathrm{e}^{-k_2 s^2},$$

where $k_1, k_2 > 0$ *are absolute constants, and* $C_1 = C_1(p, \beta)$ *and* $C_2 = C_2(p, \beta, h)$*.*

This theorem shows that the prefactor is a tight random variable, and hence constitutes a considerable sharpening of Theorem 2.2.

The proof of Theorem 2.3 uses the potential-theoretic approach to metastability.

The local homogeneity of the Erdős-Rényi random graph again plays a crucial role: it turns out that the exact same test functions and test flows that are employed in relevant variational estimates work for the Curie-Weiss model and can be used to give sharp upper and lower bounds on the average crossover time.

The better control on the prefactor comes at a price:

- The magnetic field has to be taken small enough.
- The dynamics starts from the last-exit biased distribution on $\mathbf{m}_N^-$ for the transition from $\mathbf{m}_N^-$ to $\mathbf{m}_N^+$, rather than from an arbitrary configuration in $\mathbf{m}_N^-$.

§ TECHNIQUES

Proofs rely on elaborate techniques:

isoperimetric inequalities
concentration estimates
capacity estimates
coupling techniques
coarse-graining techniques
...

These techniques exploit the fact that in the dense regime the Erdős-Rényi random graph is locally homogeneous.

§ SIM ON THE INHOMOGENEOUS ERRG

Theorem 2.3 can be extended to the inhomogeneous ERRG. The Hamiltonian becomes

$$H(\sigma) = -\sum_{(i,j)\in E} J_{ij}\sigma_i\sigma_j - h\sum_{i\in V}\sigma_i$$

with $J_{ij} > 0$ independent random variables. An example is Bernoulli with probability $r(\frac{i}{N}, \frac{j}{N})$, where

$$r(x,y), \qquad x,y \in [0,1],$$

is a continuous reference graphon. A special case is the rank-1 choice $r(x,y) = v(x)v(y)$ for some weight function $v(x)$, $x \in [0,1]$, which corresponds to the Chung-Lu Random Graph.

Bovier, den Hollander, Marello, Pulvirenti, Slowik 2022 + 2024

§ SIM ON SPARSE GRAPHS

ERRG is a dense graph. We next consider sparse graphs. Given a finite connected non-oriented multigraph

$$G = (V, E),$$

the Hamiltonian is

$$H(\sigma) = -\frac{J}{2} \sum_{(i,j) \in E} \sigma_i \sigma_j - \frac{h}{2} \sum_{i \in V} \sigma_i, \qquad \sigma \in \Omega,$$

where $J > 0$ is the ferromagnetic pair potential and $h > 0$ is the magnetic field.

We write $\mathbb{P}_\sigma^{G,\beta}$ to denote the law of $(\sigma_t^G)_{t \geq 0}$ given $\sigma_0^G = \sigma$.

The upper indices G, β exhibit the dependence on the underlying graph G and the interaction strength β between neighbouring spins.

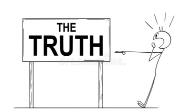
It is easy to check that $\mathbf{s} = \{\boxplus\}$ for all G because $J, h > 0$. For general G, however, $\mathbf{m}$ is not a singleton, but we will be interested in those G for which the following hypothesis is satisfied

$$\text{(H)} \qquad \mathbf{m} = \{\boxminus\}.$$

The energy barrier between $\boxminus$ and $\boxplus$ is

$$\Gamma^\star = H(\mathcal{C}^\star) - H(\boxminus),$$

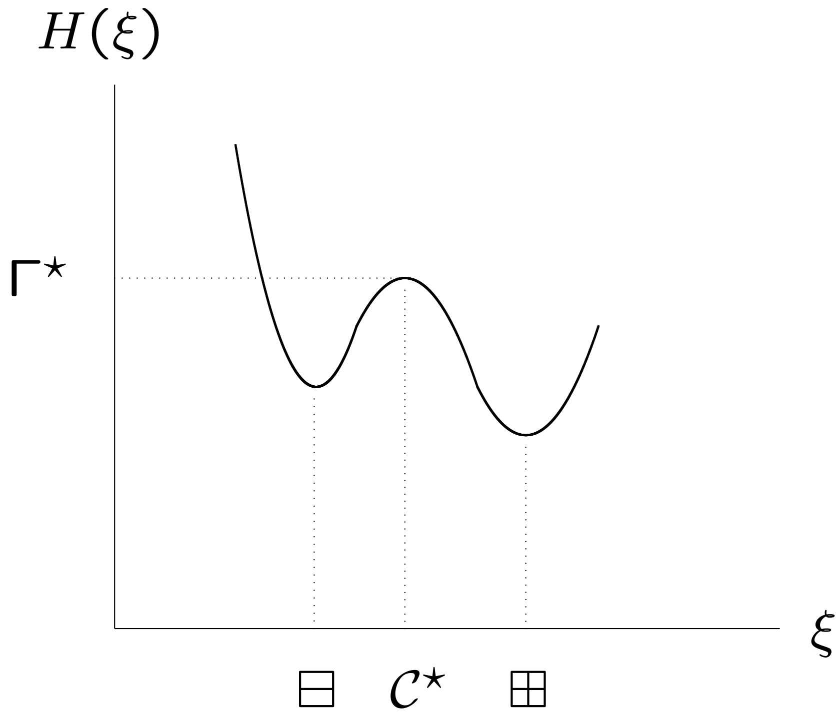
where $\mathcal{C}^\star = \mathbf{c}$ is the set of critical configurations realising the min-max for the crossover from $\boxminus$ to $\boxplus$, all of which have the same energy.



THEOREM 2.4 Bovier, den Hollander 2015

There exists a $K^\star \in (0, \infty)$*, called prefactor, such that*

$$\lim_{\beta \to \infty} \mathrm{e}^{-\beta \Gamma^\star} \mathbb{E}^{G,\beta}_{\boxminus}(\tau_{\boxplus}) = K^\star.$$



Schematic picture of H and $\boxminus, \boxplus$ and $\Gamma^\star, \mathcal{C}^\star$.

The validity of Theorem 2.4 does not rely on the details of the graph G, provided it is finite, connected and non-oriented. For concrete choices of G, the task is to identify the critical triplet

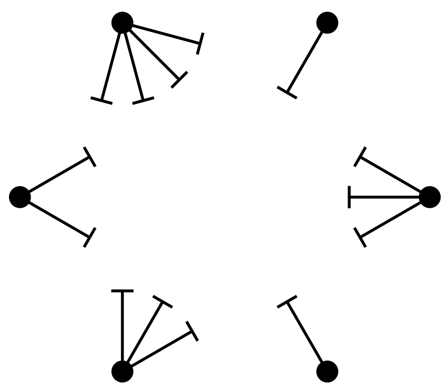
$$(\mathcal{C}^\star, \Gamma^\star, K^\star).$$

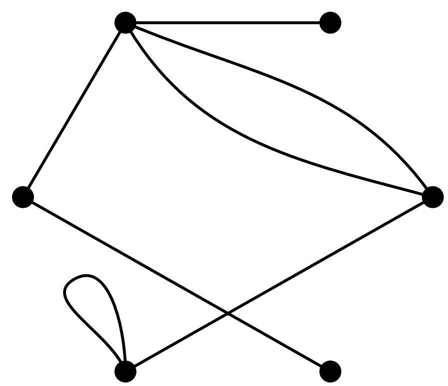
For deterministic graphs this task has been successfully carried out for a large number of examples. However, for random graphs the triple is random, and identification represents a very serious challenge.

In what follows we focus on the CM.

§ SIM ON THE CONFIGURATION MODEL

The CM is a sparse graph that can be generated via a simple pairing algorithm.





size 6
degrees $(1, 3, 1, 3, 2, 4)$
randomly pair half-edges

WARNING: The text on pages 24-34 is rather technical. We go over it in leaps to sketch the main picture.

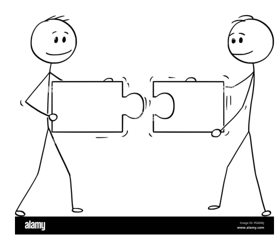
In order to state our main theorems, we need some notations and definitions.

1. Fix $N \in \mathbb{N}$. With each vertex $i \in [N]$ we associate a random degree d_i, in such a way that

$$(d_i)_{i\in[N]}$$

are i.i.d. with probability distribution f conditional on the event $\{\sum_{i\in[N]} d_i = \text{even}\}$. Consider a uniform matching of the half-edges, leading a multi-graph CM_N satisfying the requirement that the degree of vertex i is d_i for $i \in [N]$.

The total number of edges is $\frac{1}{2}\sum_{i\in[N]} d_i$.



2. Throughout the sequel we write $\mathbb{P}_N$ to denote the law of the random multi-graph CM_N generated by the Configuration Model.

3. To avoid degeneracies we assume that

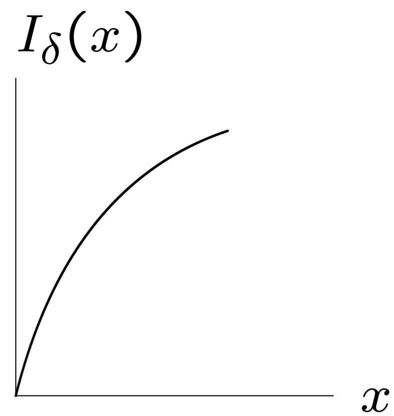
$$d_{\text{min}} = \min\{k \in \mathbb{N}\colon\ f(k) > 0\} \geq 3,$$
$$d_{\text{ave}} = \sum_{k\in\mathbb{N}} k f(k) < \infty,$$

i.e., all degrees are at least three and the average degree is finite. In this case CM_N is connected with high probability (whp), i.e., with probability tending to 1 as $N \to \infty$.

4. Along the way we need a technical function that allows us to quantify certain properties of the energy landscape, which we introduce next. Later we provide the underlying heuristics.

For $x \in (0, \frac{1}{2}]$ and $\delta \in (1, \infty)$, define

$$I_\delta(x) = \inf \Big\{ y \in (0, x] \colon \; 1 < x^{x(1-1/\delta)} (1-x)^{(1-x)(1-1/\delta)} \times (1-x-y)^{-(1-x-y)/2} (x-y)^{-(x-y)/2} y^{-y} \Big\}.$$



Plot of the function $x \mapsto I_\delta(x)$ for $\delta = 6$.

§ MAIN THEOREMS

Dommers, den Hollander, Jovanovski, Nardi 2017

We want to prove Hypothesis (H) and also to identify the critical triplet for CM_N, which we henceforth denote by $(\mathcal{C}_N^\star, \Gamma_N^\star, K_N^\star)$, in the limit as $N \to \infty$.

Our first theorem settles Hypothesis (H) for small h/J. Suppose that

$$\frac{h}{J} < \frac{2I_{d_{\text{ave}}}\left(\frac{1}{2}\right) - \frac{1}{2}\left(1 - 4I_{d_{\text{min}}}\left(\frac{1}{2}\right)\right)^2 \left(1 - 2I_{d_{\text{min}}}\left(\frac{1}{2}\right)\right)^{-1}}{\left(\frac{1}{d_{\text{ave}}} + \frac{1}{2}\right)}.$$

THEOREM 2.5.

If the above inequality is satisfied, then

$$\lim_{N\to\infty} \mathbb{P}_N\Big(\mathrm{CM}_N \text{ satisfies (H)}\Big) = 1.$$

Our second and third theorem provide upper and lower bounds on $\Gamma^\star_N$. Label the vertices of the graph in order of increasing degree. Let $\gamma\colon \boxminus \to \boxplus$ be the path that successively flips the vertices $1,\dots,N$ (in that order), and for $M \in [N]$ let $\ell_M = \sum_{i\in[M]} d_i$.

THEOREM 2.6

Define

$$\begin{aligned}\bar{M} &= \bar{M}\left(\frac{h}{J}\right)\\ &= \min\left\{M \in [N]\colon\ \frac{h}{J} \geq \ell_{M+1}\left(1-\frac{\ell_{M+1}}{\ell_N}\right) - \ell_M\left(1-\frac{\ell_M}{\ell_N}\right)\right\},\end{aligned}$$

and note that $\bar{M} < N/2$. *Then with high probability*

$$\Gamma^\star_N \leq \Gamma^+_N, \qquad \Gamma^+_N = J\ell_{\bar{M}}\left(1-\frac{\ell_{\bar{M}}}{\ell_N}\right) - h\bar{M} \pm O\left(\ell_N^{3/4}\right).$$

THEOREM 2.7

Define

$$\tilde{M} = \min\left\{M \in [N]\colon\ \ell_M \geq \tfrac{1}{2}\ell_N\right\}.$$

Then whp

$$\Gamma^\star_N \geq \Gamma^-_N, \qquad \Gamma^-_N = J\, d_{\mathrm{ave}}\, I_{d_{\mathrm{ave}}}\left(\tfrac{1}{2}\right) N - h\tilde{M} - o(N).$$

COROLLARY 2.8

Under Hypothesis (H), *Theorems 2.6–2.7 yield*

$$\lim_{\beta\to\infty} \mathbb{P}^{G,\beta}_{\boxminus}\left(\mathrm{e}^{\beta(\Gamma^-_N-\varepsilon)} \leq \tau_{\boxplus} \leq \mathrm{e}^{\beta(\Gamma^+_N+\varepsilon)}\right) = 1 \quad \forall\, \varepsilon > 0.$$

REMARK: For simple degree distributions, like Dirac or power law, the quantities $\bar{M}$, $\ell_{\bar{M}}$, $\tilde{M}$ can be computed explicitly.

The bounds in Theorems 2.6–2.7 are tight in the limit of large degrees. Indeed, by the law of large numbers we have that

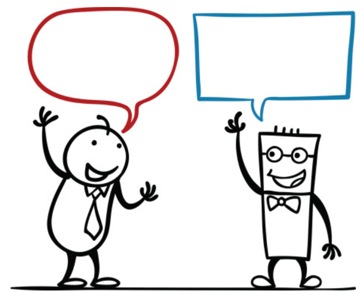
$$\ell_N \frac{\ell_{\bar{M}}}{\ell_N}\left(1-\frac{\ell_{\bar{M}}}{\ell_N}\right) \le \tfrac{1}{4}\ell_N = \tfrac{1}{4} d_{\mathsf{ave}}\, N\,[1+o(1)]\,.$$

Hence

$$\frac{\Gamma_N^+}{\Gamma_N^-} = \frac{\tfrac{1}{4} d_{\mathsf{ave}}\,[1+o\,(1)] - \frac{h}{J}\frac{\bar{M}}{N} + o(1)}{d_{\mathsf{ave}} I_{d_{\mathsf{ave}}}\left(\frac{1}{2}\right) - \frac{h}{J}\frac{\tilde{M}}{N} - o(1)}.$$

In the limit as $d_{\mathsf{ave}} \to \infty$ we have $I_{d_{\mathsf{ave}}}\left(\frac{1}{2}\right) \to \frac{1}{4}$, in which case the above ratio tends to 1.

§ DISCUSSION



1. The integer $\bar{M}$ has the following interpretation. The path $\gamma\colon \boxminus \to \boxplus$ is obtained by flipping (-1)-valued vertices to $(+1)$-valued vertices in order of increasing degree. Up to fluctuations of size $o(N)$, the energy along γ increases for the first $\bar{M}$ steps and decreases for the remaining $N - \bar{M}$ steps.

2. The integer $\tilde{M}$ has the following interpretation. To obtain our lower bound on $\Gamma_N^\star$ we consider configurations whose $(+1)$-valued vertices have total degree at most $\frac{1}{2}\ell_N$. The total number of $(+1)$-valued vertices in such type of configurations is at most $\tilde{M}$.

3. If we consider all sets on CM_N that are of total degree $x\ell_N$ and share $y\ell_N$ edges with their complement, then $I_\delta(x)$ represents (a lower bound on) the least value for y such that the average number of such sets is at least 1. In particular, for smaller values of y this average number is exponentially small.

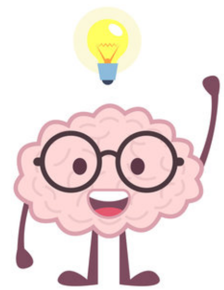
4. We believe that Hypothesis (H) holds as soon as

$$0 < h < (d_{\mathsf{min}} - 1)J,$$

i.e., we believe that in the limit as $\beta \to \infty$ followed by $N \to \infty$ this choice of parameters corresponds to the metastable regime of our dynamics, i.e., the regime where $(\boxminus, \boxplus)$ is a metastable pair.

5. The scaling behaviour of $\Gamma_N^\star, K_N^\star$ as $N \to \infty$, as well as the geometry of $\mathcal{C}_N^\star$, are hard to capture. Here are some conjectures.

Dommers, den Hollander, Jovanovski, Nardi 2017



CONJECTURE 2.9

There exists a $\gamma^\star \in (0,\infty)$ *such that*

$$\lim_{N\to\infty} \mathbb{P}_N\left(\left|N^{-1}\Gamma_N^\star - \gamma^\star\right| > \delta\right) = 0 \qquad \forall\, \delta > 0.$$

CONJECTURE 2.10

There exists a $c^\star \in (0,1)$ *such that*

$$\lim_{N\to\infty} \mathbb{P}_N\left(\left|N^{-1}\log|\mathcal{C}_N^\star| - c^\star\right| > \delta\right) = 0 \qquad \forall\, \delta > 0.$$

CONJECTURE 2.11

There exists a $\kappa^\star \in (1,\infty)$ *such that*

$$\lim_{N\to\infty} \mathbb{P}_N\left(\left||\mathcal{C}_N^\star|\, K_N^\star - \kappa^\star\right| > \delta\right) = 0 \qquad \forall\, \delta > 0.$$

6. It is shown in Dommers 2017 that for a random regular graph with degree $r \geq 3$, there exist constants $0 < \gamma_-^\star(r) < \gamma_+^\star(r) < \infty$ such that

$$\lim_{N\to\infty} \lim_{\beta\to\infty} \mathbb{E}_N \left(\mathbb{P}_{\boxminus}^{\mathrm{CM}_N} \left(\mathrm{e}^{\beta N \gamma_-^\star(r)} \leq \tau_{\boxplus} \leq \mathrm{e}^{\beta N \gamma_+^\star(r)} \right) \right) = 1$$

when $\frac{h}{J} \in (0, C_0\sqrt{r})$ for some constant $C_0 \in (0,\infty)$ that is small enough.

Moreover, there exist constants $C_1 \in (0, \frac{1}{4}\sqrt{3})$ and $C_2 \in (0,\infty)$ (depending on C_0) such that

$$\gamma_-^\star(r) \geq \tfrac{1}{4}Jr - C_1 J\sqrt{r}, \qquad \gamma_+^\star(r) \leq \tfrac{1}{4}Jr + C_2 J\sqrt{r}.$$

These results are derived without Hypothesis (H), but it is shown that Hypothesis (H) holds as soon as $r \geq 6$.