LECTURE 3

The Voter Model (VM)

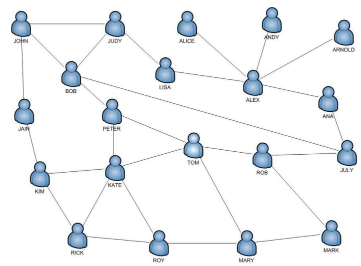
§ TARGET

In this lecture we focus on the VM on the regular random graph. We analyse how the fraction of discordant edges evolves over time, in the limit as the size of the graph tends to infinity, on three time scales: short, moderate, and long.

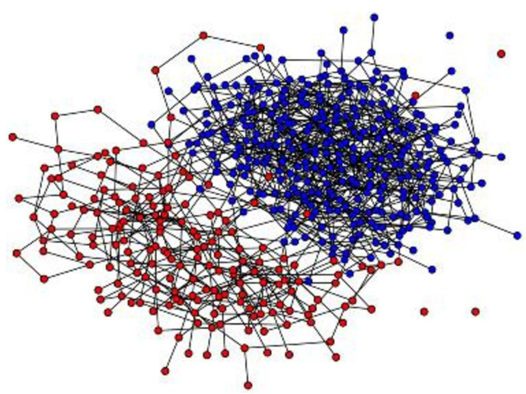
We also analyse what happens when the edges of the random regular graph are randomly rewired while the VM is running. It will turn out that the graph dynamics has several interesting consequences.

Avena, Baldasso, Hazra, den Hollander, Quattropani 2022+2024

SOCIAL NETWORK:



VOTER MODEL:



Given a connected graph $G = (V, E)$, the voter model is the Markov process $(\xi_t)_{t\geq 0}$ on state space $\{0,1\}^V$ where each vertex carries opinion 0 or 1, at rate 1 selects one of the neighbouring vertices uniformly at random, and adopts its opinion.

Write $\xi_t = \{\xi_t(i)\colon\ i \in V\}$ with $\xi_t(i)$ the opinion at time t of vertex i. We analyse the evolution of the fraction of discordant edges

$$\mathcal{D}_t^N = \frac{|D_t^N|}{M}, \qquad D_t^N = \Big\{(i,j) \in E\colon\ \xi_t(i) \neq \xi_t(j)\Big\},$$

where $N = |V|$ and $M = |E|$. This is an interesting quantity because it monitors the size of the boundary between the two opinions.

The consensus time is defined as

$$\tau_{\text{cons}} = \inf\{t \geq 0 \colon \xi_t(i) = \xi_t(j)\,\forall\, i,j \in V\}.$$

For finite graphs we know that $\tau_{\text{cons}} < \infty$ with probability 1, either at $[0]_N$ or at $[1]_N$. The interest lies in determining the relevant time scales on which consensus is reached, and how it is reached.

Via time reversal, the voter model is dual to a system of coalescing random walks, describing the genealogy of the opinions.

§ VM ON THE COMPLETE GRAPH

As a prelude we look at the VM on the complete graph, for which computations can be carried through explicitly. Indeed, the number of 1-opinions at time t, given by

$$O_t^N = \sum_{i \in V} \xi_t(i),$$

performs a continuous-time nearest-neighbour random walk on the set $\{0, \ldots, N\}$ with transition rates

$$\begin{aligned} n \to n+1 &\quad \text{at rate } n(N-n)\tfrac{1}{N-1}, \\ n \to n-1 &\quad \text{at rate } (N-n)n\tfrac{1}{N-1}. \end{aligned}$$

This is the same as the Moran model from population genetics.

Put $\mathcal{O}_t^N = \frac{1}{N} O_t^N$ for the fraction of 1-opinions at time t. Then it is well-known that

$$(\mathcal{O}_{sN}^N)_{s \geq 0}$$

converges in law as $N \to \infty$ to the Fisher-Wright diffusion $(\chi_s)_{s \geq 0}$ on $[0, 1]$ given by

$$\mathrm{d}\chi_s = \sqrt{2\chi_s(1-\chi_s)}\, \mathrm{d}W_s,$$

where $(W_s)_{s \geq 0}$ is standard Brownian motion.

EXERCISE Give the proof of the above convergence.

The number of discordant edges equals

$$D_t^N = \frac{O_t^N(N - O_t^N)}{2}.$$

Recall that $\mathcal{D}_t^N = \frac{1}{M} D_t^N$ denotes the fraction of discordant edges at time t, with $M = \binom{N}{2}$ for the complete graph.

Since

$$\mathcal{D}_t^N = \frac{O_t^N(N - O_t^N)}{N(N-1)} = \frac{N}{N-1}\,\mathcal{O}_t^N(1-\mathcal{O}_t^N),$$

it follows that

$$(\mathcal{D}_{sN}^N)_{s\geq 0}$$

converges in law as $N \to \infty$ to the process

$$\Big(\chi_s(1-\chi_s)\Big)_{s\geq 0}.$$

In the mean-field setting of the complete graph, the fraction of discordant edges is the product of the fractions of the two opinions.

The latter property fails on non-complete graphs, in particular, on random graphs.

§ VM ON THE RANDOM REGULAR GRAPH

Consider the regular random graph $G_{d,N} = (V, E)$ of degree $d \geq 3$, consisting of

$$|V| = N \text{ vertices},$$
$$|E| = M = \frac{dN}{2} \text{ edges}.$$

Denote the law of $G_{d,N}$ by $\mathbb{P}$.

Chen, Choi, Cox 2016 consider the fraction of 1-opinions at time t,

$$\mathcal{O}_t^N = \frac{1}{N} \sum_{i \in V} \xi_t(i),$$

and show that

$$(\mathcal{O}_{sN}^N)_{s \geq 0}$$

converges in law as $N \to \infty$ to the Fisher-Wright diffusion $(\chi_s)_{s \geq 0}$ given by

$$\mathrm{d}\chi_s = \sqrt{2\theta_d \chi_s (1 - \chi_s)}\, \mathrm{d}W_s,$$

where $(W_s)_{s \geq 0}$ is standard Brownian motion, and

$$\theta_d = \frac{d-2}{d-1}.$$

EXERCISE Use duality and the graphical representation to write the probability that a fixed edge is discordant at time t in terms of the meeting time of random walks.

§ MAIN THEOREMS

For $u \in (0,1)$, let $\mathbf{P}_u$ be the law of $(\mathcal{D}_t^N)_{t \geq 0}$ starting from $[\text{Bernoulli}(u)]^N$.

THEOREM 3.1 Mean fraction on arbitrary time scales

Fix $u \in (0,1)$. Then, for any $t_N \in [0,\infty)$,

$$\left| \mathbf{E}_u \left[\mathcal{D}_{t_N}^N \right] - 2u(1-u)\, f_d(t_N)\, \mathrm{e}^{-2\theta_d \frac{t_N}{N}} \right| \xrightarrow{\mathbb{P}} 0,$$

where

$$f_d(t) = \mathbf{P}^{\mathcal{T}_d}(\tau_{\text{meet}} > t),$$

with $\mathbf{P}^{\mathcal{T}_d}$ the law of two independent random walks on the infinite d-regular tree $\mathcal{T}_d$ starting from the endpoints of an edge.

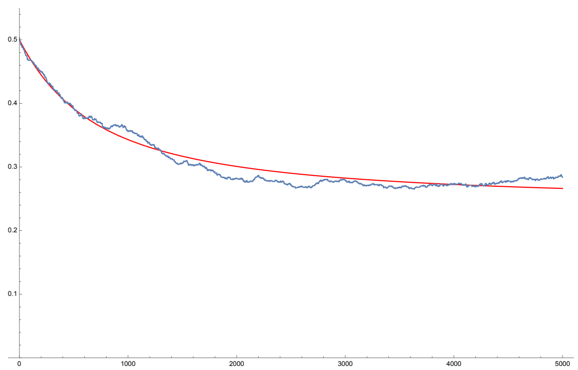
The profile function f_d is given by

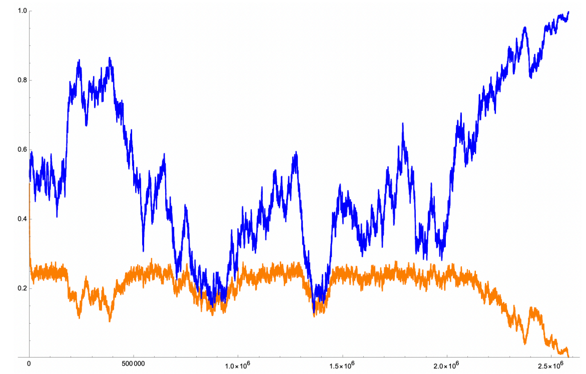
$$f_d(t) = \sum_{k=0}^{\infty} \mathrm{e}^{-2t} \frac{(2t)^k}{k!} \sum_{l > \lfloor \frac{k-1}{2} \rfloor} \binom{2l}{l} \frac{1}{l+1} \left(\frac{1}{d}\right)^{l+1} \left(\frac{d-1}{d}\right)^{l},$$

and satisfies $f_d(0) = 1$ and $f_d(\infty) = \theta_d$.

Note that Theorem 3.1 shows three times scales:

short:	$t_N \ll N$,
moderate:	$t_N \asymp N$,
long:	$t_N \gg N$.

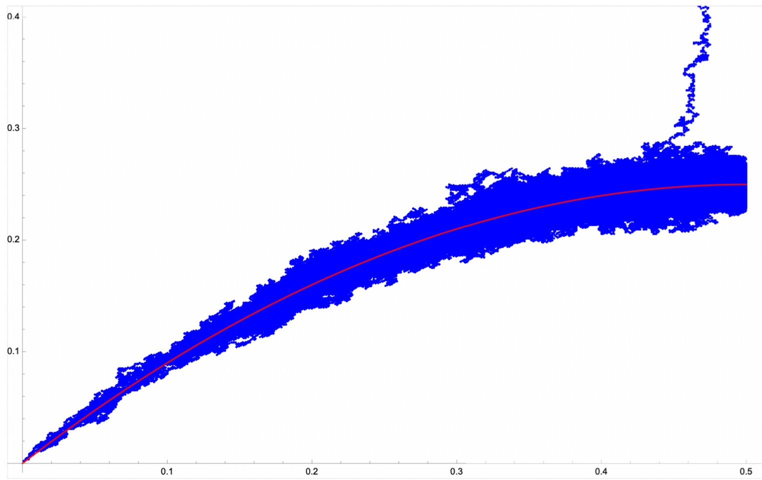




A single simulation for $N = 1000$, $d = 3$, $u = 0.5$.

Left: In blue the fraction of discordant edges up to $t = 5$, in red the function $t \mapsto 2u(1-u)\, f_d(t)$.

Right: In blue the fraction of 1-opinions up to consensus, in orange the fraction of discordant edges up to consensus.



Scatter plot for the same simulation: the fraction of discordant edges versus the fraction of the minority opinion.

The piece sticking out corresponds to short times. The curve in red is $x \mapsto x(1-x)$, which says that the fraction of discordant edges is close to the product of the fractions of the two opinions.

THEOREM 3.2

Concentration on short time scales and scaling on moderate time scales

(i) *Let* t_N *be such that* $t_N/N \to 0$. *Then, for every* $\varepsilon > 0$,

$$\sup_{\eta \in \{0,1\}^V} \mathbf{P}_\eta \left(\left| \mathcal{D}^N_{t_N} - \mathbf{E}_\eta[\mathcal{D}^N_{t_N}] \right| > \varepsilon \right) \xrightarrow{\mathbb{P}} 0.$$

(ii) *Let* t_N *be such that* $t_N/N \to s \in (0,\infty)$. *Then, for every* $u \in (0,1)$,

$$\sup_{x \in [0,1]} \left| \mathbf{P}_u \left(\mathcal{D}^N_{t_N} \le x \right) - \mathbf{P}_u \left(\chi_s (1-\chi_s) \le x \right) \right| \xrightarrow{\mathbb{P}} 0.$$

THEOREM 3.3 Uniform concentration on short time scales

Fix $u \in (0,1)$. *Then, for every* $\delta, \epsilon > 0$,

$$\mathbf{P}_u\Big(\sup_{0 \le t \le N^{1-\delta}} \Big| \mathcal{D}_t^N - \mathbf{E}_u[\mathcal{D}_t^N] \Big| > \varepsilon \Big) \xrightarrow{\mathbb{P}} 0.$$

§ OPEN PROBLEMS

- We expect that Theorems 3.1–3.2 can be extended to non-regular sparse random graphs. We do not have a conjecture on how the function f_d and the diffusion constant θ_d modify in this more general setting.

- We expect that Theorem 3.3 can be strengthened to the statement that, for every $u \in (0,1)$, every t_N such that $t_N/N \to 0$ and every $C_N \to \infty$,

$$\mathbf{P}_u\Big(\big|\mathcal{D}^n_{t_N} - \mathbf{E}_u[\mathcal{D}^N_{t_N}]\big| > C_N \sqrt{t_N/N} \Big) \xrightarrow{\mathbb{P}} 0.$$

For directed sparse random graphs more can be said.

THEOREM 3.4

Avena, Capannoli, Hazra, Quattropani 2023, Capannoli 2024

Under mild conditions on the in-degrees $d^{\text{in}} = (d_i^{\text{in}})_{i=1}^N$ *and the out-degrees* $d^{\text{out}} = (d_i^{\text{out}})_{i=1}^N$*, the same scaling applies and an explicit formula can be derived for the profile function* f_d *and the diffusion constant* $\theta_{d^{\text{in}},d^{\text{out}}}$*.*

For instance, if $d^{\text{in}} = d^{\text{out}}$ (= Eulerian graph), then

$$\theta_{d^{\text{in}},d^{\text{out}}} = \left(\frac{m_2}{m_1^2} - 1 + \sqrt{1 - \frac{1}{m_1}}\right)^{-1}$$

with m_1, m_2 the first and the second moment of the limit of the empirical degree distribution.

§ IDEAS IN THE PROOF OF THE MAIN THEOREMS

The proofs are based on the classical notion of duality between the voter model and a collection of coalescent random walks.

A crucial role is played by properties of coalescing random walks that hold in mean-field geometries. In particular, Oliveira 2013

$$\lim_{N\to\infty} \frac{\mathbb{E}[\tau_{\text{coal}}]}{\mathbb{E}[\tau_{\text{meet}}^{\pi\otimes\pi}]} = 2,$$

where τ_{coal} is the coalescence time of N random walks, each starting from a different vertex, while $\tau_{\text{meet}}^{\pi\otimes\pi}$ is the meeting time of two random walks, independently starting from the stationary distribution π.

EXERCISE Explain the above result heuristically. Think about what happens on the complete graph.

1. On time scales $o(\log N)$, below the typical distance between two vertices, the analysis proceeds by coupling two random walks on the d-regular random graph with two random walks on the d-regular tree, both starting from adjacent vertices.

Because the tree is regular, the distance of the two random walks can be viewed as the distance to the origin of a single biased random walk on $\mathbb{N}_0$ starting from 1. Note that the same does not hold when the tree is not regular.

EXERCISE

Compute τ_{meet} on a Galton-Watson tree by reinterpreting the problem as a single biased random walk on $\mathbb{N}_0$.

2. On time scale $\Theta(\log N)$, the scale of the typical distance between two vertices, the coupling argument is combined with a finer control of the two random walks on the d-regular random graph.

LEMMA 3.5

There is a sequence of random variables $(\theta_{d,N})_{N\in\mathbb{N}}$ *converging to* θ_d *such that*

$$\lim_{N\to\infty} \sup_{t\geq 0} \left| \frac{\mathbb{P}(\tau^{\pi\otimes\pi}_{\text{meet}} > t)}{\exp[-2\theta_{d,N}(t/N)]} - 1 \right| = 0 \quad \text{in probability.}$$

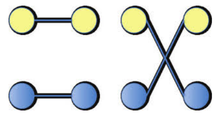
Lemma 3.4, together with a first-moment argument, is enough to compute the evolution of the expected number of discordant edges on every time scale.

3. In order to obtain concentration, a much deeper analysis is required. Roughly, in order to have proper control on the correlations between the discordant edges, we must analyse a dual system of random walks whose number grows with N.

An upper bound is derived for the number of meetings of a poly-logarithmic number of independent random walks evolving on the random graph for a time $N^{1-o(1)}$.

This is exploited to derive an upper bound for the deviation from the mean that is exponentially small in N and uniform in time. This upper bound can be translated into a concentration estimate by taking a union bound.

§ REWIRING



What happens when the graph itself evolves over time, e.g. the edges are randomly rewired?

Suppose that every pair of edges swaps endpoints at rate $\nu/2M$ with $\nu \in (0,\infty)$. With this choice of parametrization, the rate at which a given edge is involved in a rewiring converges to ν as $N \to \infty$. The voter model on this dynamic random graph evolves as before: at rate 1 opinions are adopted along the edges that are currently present.

In work in progress we show that Theorems 3.1-3.2 carry over with θ_d replaced by $\theta_{d,\nu}$ given by a continued fraction.

THEOREM 3.6 work in progress

Let $\beta_d = \sqrt{d-1}$ *and* $\rho_d = \frac{2}{d}\sqrt{d-1}$. *Then*

$$\theta_{d,\nu} = 1 - \frac{\Delta_{d,\nu}}{\beta_d}$$

with

$$\Delta_{d,\nu} = \frac{1\,|}{|\,\frac{2+\nu}{\rho_d}} - \frac{1\,|}{|\,\frac{2+2\nu}{\rho_d}} - \frac{1\,|}{|\,\frac{2+3\nu}{\rho_d}} - \cdots$$

EXERCISE

Compute $\lim_{\nu\to\infty} \theta_{d,\nu}$ for fixed d.

1. Since the d-regular random graph locally looks like a d-regular tree, the proof proceeds by analysing the meeting time of two random walks on a d-regular tree. On short to modest time scales the two random walks do not notice the difference. Work is needed to show that on longer time scales the approximation is still good.

2. We replace rewiring of edges on the d-regular random graph by disappearance of edges on the d-regular tree. This is a good approximation because, as soon as one random walk moves along a rewired edge in the d-regular random graph, it is thrown far away from the other random walk and meeting becomes difficult.

KEY OBSERVATION:

Because $\nu \mapsto \theta_{d,\nu}$ is strictly increasing, the dynamics speeds up consensus.