LECTURE 4

The Contact Process (CP)

§ CP ON A GRAPH

Given a connected graph $G = (V, E)$, the contact process is the Markov process $(\xi_t)_{t \geq 0}$ on state space $\{0, 1\}^V$ where each vertex is either healthy (0) or infected (1).

Each infected vertex becomes healthy at rate 1, independently of the state of the other vertices, while each healthy vertex becomes infected at rate λ times the number of infected neighbours, with $\lambda \in (0, \infty)$ the infection rate.

The configuration at time t is $\xi_t = \{\xi_t(i) \colon i \in V\}$, with $\xi_t(i)$ the state at time t of vertex i.

In what follows we will analyse the behaviour of the CP on various classes of graphs, both random and deterministic. Our focus will be on understanding how the extinction time

$$\tau_{[0]_N} = \text{inf}\{t \geq 0 : \xi_t(i) = 0 \ \forall\, i \in V\}$$

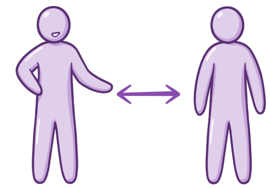
behaves as $|V| = N \to \infty$, depending on the value of λ and the properties of the graph. We will mostly zoom in on

$$\mathbb{E}_{\tau_{[1]_N}}(\tau_{[0]_N}),$$

the average extinction time starting from the configuration where every vertex is infected.

We will see that it is hard to get control on this quantity, so we will have to content ourselves with rough bounds.

§ CP ON THE COMPLETE GRAPH



As a prelude we look at the CP on the complete graph, for which computations can be carried through explicitly. Indeed, the number of infections at time t, given by

$$I_t^N = \sum_{i \in V} \xi_t(i),$$

evolves as a continuous-time nearest-neighbour random walk on the set $\{0, \ldots, N\}$ with transition rates

$$\begin{aligned} n \to n+1 &\quad \text{at rate } \lambda n(N-n), \\ n \to n-1 &\quad \text{at rate } n. \end{aligned}$$

Put $\mathcal{I}_t^N = \frac{1}{N} I_t^N$ for the fraction of infected vertices at time t. This process is a continuous-time nearest-neighbour random walk on the set

$$\left\{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\right\}$$

with transition rates

$$\begin{aligned} x &\to x + N^{-1} \quad \text{at rate } \lambda x(1-x)N^2, \\ x &\to x - N^{-1} \quad \text{at rate } xN. \end{aligned}$$

This process has a strong drift upward, which becomes zero for $x = 1 - \frac{1}{\lambda N}$, i.e., very close to full infection when $\lambda N \gg 1$.

THEOREM 4.1

Let $\tau_{[0]_N} = \inf\{t \geq 0 \colon \xi_t = [0]_N\}$ *be the extinction time. Then*

$$\log \mathbb{E}_{[1]_N}(\tau_{[0]_N}) = N[1 + \log(\lambda N)] + o(N), \quad N \to \infty.$$

Thus, the CP on the complete graph is supercritical for all $\lambda > 0$ as $N \to \infty$.

THEOREM 4.2

For every $\lambda \in (0,\infty)$,

$$\lim_{N\to\infty} \mathbb{P}_{[1]_N}\left(\frac{\tau_{[0]_N}}{\mathbb{E}_{[1]_N}(\tau_{[0]_N})} > t\right) = \mathrm{e}^{-t} \quad \forall\, t > 0.$$

metastability

In the mean-field setting of the complete graph, the fraction of infected vertices performs a random walk.

The latter property fails on non-complete graphs, in particular, on random graphs: $(\mathcal{I}_t^N)_{t\geq 0}$ loses the Markov property.

The CP is harder than the VM because it does not have a tractable dual. In fact, it is self-dual.

§ CP ON THE CONFIGURATION MODEL

Consider CP on CM with an empirical degree distribution f_N satisfying $\lim_{N\to\infty} \|f_N - f\|_\infty = 0$ with

$$f(k) = k^{-\tau+o(1)}, \qquad k \to \infty,$$

where $\tau \in (1, \infty)$ is the tail exponent.

THEOREM 4.3

Chatterjee, Durrett 2009, Mountford, Mourrat, Valesin, Yao 2016

If $\tau \in (2, \infty)$, then for every $\lambda \in (0, \infty)$ the average time to extinction grows exponentially fast with N whp.

> Thus, CP on CM with a power law degree distribution is supercritical regardless of the value of λ. Apparently, hubs easily transmit the infection.

THEOREM 4.4 Can, Schapira 2015

The same is true for $\tau \in (1,2]$, even though local convergence breaks down.

For the CP on the CM with a power law degree distribution there is anomalous scaling of the density of infections $\rho(\lambda)$ as $\lambda \downarrow 0$, namely,

$$\rho(\lambda) \asymp \begin{cases} \lambda^{1/(3-\tau)}, & \tau \in (2, \frac{5}{2}], \\ \lambda^{2\tau-3} \, [\log(1/\lambda)]^{-(\tau-2)}, & \tau \in (\frac{5}{2}, 3], \\ \lambda^{2\tau-3} \, [\log(1/\lambda)]^{-2(\tau-2)}, & \tau \in (3, \infty). \end{cases}$$

Mountford, Valesin, Yao 2013, Linker, Mitsche, Schapira, Valesin 2020

The three regimes reflect different optimal strategies to survive extinction for small infection rates: the infection survives close to hubs.

Sharp estimates of the extinction time have been obtained for the CP on the CM with i.i.d. degrees $(D_i)_{i=1}^N$ taking values in $\mathbb{N}_0$. When $\mathbb{E}[D_1] < \infty$ we expect a strictly positive critical threshold.

THEOREM 4.5 Cator, Don 2021

Suppose that $\mathbb{E}[D_1] < \infty$ *and* $\mathbb{E}[2^{-D_1/2}] < \frac{1}{4}$. *Then there exists a constant* $\alpha \in (0, \mathbb{E}[D_1]]$ *such that if* $\lambda > 1/\alpha$, *then there exists a constant* $c > 0$ *such that*

$$\mathbb{E}_{[1]_N}(\tau_{[0]_N}) \geq \mathrm{e}^{cN} \text{ whp.}$$

For the random regular graph with degree $d \geq 3$, the claim holds with $\alpha = d - 2$.

§ CP ON THE ERDŐS-RÉNYI RANDOM GRAPH

Let $p = p_N$ be the edge retention probability and $\lambda = \lambda_N$ the infection rate.

THEOREM 4.6 Cator, Don 2021

(a) *If* $\lim_{N\to\infty} Np_N = \infty$, *then*

$$\mathbb{E}_{[1]_N}(\tau_{[0]_N}) \geq \mathrm{e}^{c_N N} \text{ whp}$$

for $c_N < \log(Np_N\lambda_N) - \frac{1}{Np_N\lambda_N} + 1$ *and* N *large enough.*

(b) *If* $\lim_{N\to\infty} Np_N = \sigma \in (4\log 2, \infty)$, *then*

$$\mathbb{E}_{[1]_N}(\tau_{[0]_N}) \geq \mathrm{e}^{cN} \text{ whp}$$

for $\lambda > 1/\sigma$ *and* $c < \log(\sigma\lambda) - \frac{1}{\sigma\lambda} + 1$ *and* N *large enough.*

§ CP ON THE PREFERENTIAL ATTACHMENT MODEL

THEOREM 4.7

Berger, Borgs, Chayes, Saberi 2014

Can 2015

There exists a $c > 0$ *such that*

$$\log \mathbb{E}_{[1]_N}(\tau_{[0]_N}) \geq c \frac{\lambda^2 N}{(\log N)^{\frac{1+\gamma}{1-\gamma}}}$$

for $\lambda > 0$ *small enough and* N *large enough, with* $\gamma \in [0, 1)$ *a parameter that controls the attachment probabilities of newly incoming vertices.*

§ CP ON TREE-LIKE RANDOM GRAPHS

Many sparse random graphs are locally tree-like, and hence it is interesting to study the extinction time of the CP on regular trees. Let

$$0 < \lambda_d < \infty$$

be the critical threshold for survival on the d-regular tree.

THEOREM 4.8 Mourrat, Valesin 2016

Let $\mathcal{G}_{d,N}$ be the class of connected graphs with N vertices and maximal degree d. Then, for $\lambda < \lambda_d$,

$$\lim_{N\to\infty} \sup_{G\in\mathcal{G}_{d,N}} \mathbb{P}_{[1]_G}(\tau_{[0]_G} > c \log N) = 0,$$

for some $c = c(\lambda, d)$.

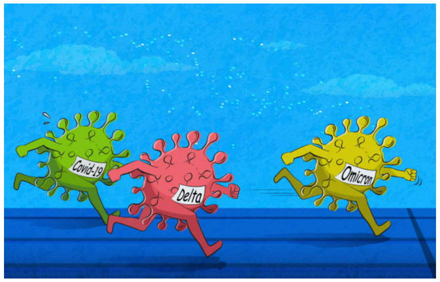
THEOREM 4.9 Mourrat, Valesin 2016

On the d-regular random graph with $d \geq 3$, the crossover from logarithmic to exponential extinction time occurs at λ_d.

THEOREM 4.10 Baptista da Silva, Oliveira, Valesin 2022

On the dynamic d-regular random graph with $d \geq 3$ and rewiring rate $\nu > 0$, the crossover occurs at a strictly smaller value than λ_d.

The rewiring helps the infection spread more easily.



§ CP ON GENERAL FINITE GRAPHS

What can be said about the extinction time for the CP on general finite graphs?

THEOREM 4.11 Mountford, Mourrat, Valesin, Yao 2016

For any $\lambda > \lambda_1$, any $D \in \mathbb{N}$ and any connected graph G whose degrees are bounded by D,

$$\mathbb{E}_{[1]_G}(\tau_{[0]_G}) \geq \exp\left[c|V|\right]$$

for some $c = c(\lambda, D) > 0$.

THEOREM 4.12 Schapira, Valesin 2017

For any $\lambda > \lambda_1$, any $\epsilon > 0$ and any connected graph G,

$$\mathbb{E}_{[1]_G}(\tau_{[0]_G}) \geq \exp\left[\frac{c|V|}{(\log|V|)^{1+\epsilon}}\right]$$

for some $c = c(\epsilon) > 0$.

CONCLUSION:

For the CP it is hard to get sharp control on the extinction time, and many questions remain open.